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Cycles, Eulerian Digraphs, and the Schönemann-Gauss Theorem

For compact maps f in normed spaces and their iterates, one has under very natural conditions the following congruences for the Leray-Schauder degrees in the case of prime numbers p [9, 6, 7]

$$\text{(1)} \qquad \mathrm{Deg}(Id - f^p, U) \equiv \mathrm{Deg}(Id - f, U) \pmod{p}$$

and of general $n \in \mathbb{N}$ [10]

$$\text{(2)} \qquad \sum_{d|n} \mu(\frac{n}{d}) \cdot \mathrm{Deg}(Id - f^d, U) \equiv 0 \pmod{n},$$

where μ is the Möbius function

$$\mu(r) := \begin{cases} 1 & \text{if } r = 1, \\ 0 & \text{if } r \text{ is divisible by the square of an integer } \geq 2, \\ (-1)^l & \text{if } r \text{ is a product of } l \text{ different prime numbers.} \end{cases}$$

The close relations with Fermat's little theorem and its generalization by Gauss are even more apparent if we look at the corresponding congruences for the Lefschetz numbers $\lambda(\cdot)$ for maps g and their iterates

$$\lambda(g^p) \equiv \lambda(g) \pmod{p} \quad \text{and} \quad \sum_{d|n} \mu(\frac{n}{d}) \cdot \lambda(g^d) \equiv 0 \pmod{n},$$

which, in the case of a space with finitely generated homology, can be derived from the congruences for integer $m \times m$ matrices A [1, 4, 8]

$$\text{(3)} \qquad \mathrm{Trace}(A^p) \equiv \mathrm{Trace}(A) \pmod{p}$$

and

$$\text{(4)} \qquad \sum_{d|n} \mu(\frac{n}{d}) \cdot \mathrm{Trace}(A^d) \equiv 0 \pmod{n}.$$

(3) is part of the classical result of Schönemann [5]

$$\text{(5)} \qquad a_j^{(p)} \equiv a_j^{(1)} \pmod{p} \quad \text{for all} \quad j = 0, \ldots, m-1$$

for the characteristic polynomials

$$P_d(x) := x^m + a_{m-1}^{(d)} x^{m-1} + \ldots + a_0^{(d)} := \det(xI - A^d), \quad d \in \mathbb{N},$$

since $a_{m-1}^{(d)} = -\mathrm{Trace}(A^d)$.